Anomalous Scaling and Refined Similarity of an Active Scalar in a Model of Homogeneous Turbulent Convection

Emily S. C. Ching[1,2] and W.C. Cheng[1]
[1] *Department of Physics, The Chinese University of Hong Kong, Shatin, Hong Kong and*
[2] *Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, Hong Kong*

(Dated: February 1, 2008)

Anomalous scaling in the statistics of an active scalar in homogeneous turbulent convection is studied using a dynamical shell model. We extend refined similarity ideas for homogeneous and isotropic turbulence to homogeneous turbulent convection and attribute the origin of the anomalous scaling to variations of the entropy transfer rate. We verify the consequences and thus the validity of our hypothesis by showing that the conditional statistics of the active scalar and the velocity at fixed values of entropy transfer rate are not anomalous but have simple scaling with exponents given by dimensional considerations, and that the intermittency corrections are given by the scaling exponents of the moments of the entropy transfer rate.

PACS numbers: 47.27.-i, 47.27.eb

Since the work of Kolmogorov in 1941 (K41) [1], much effort has been devoted to the study of the possible universal statistics of fluid turbulence in the inertial range, the range of length scales that are smaller than those of energy input and larger than those directly affected by molecular dissipation. A major challenge is to understand, from first principles, the origin of anomalous scaling, which is the deviation of the velocity scaling behavior from those predicted by dimensional considerations in K41. One important idea proposed by Kolmogorov in his refined theory [2], which we refer to as Kolmogorov's refined similarity idea, attributes the origin of anomalous scaling of the velocity to the variations of local energy dissipation rate. Kraichnan [3] later pointed out that the local energy dissipation rate is not an inertial-range quantity and proposed to attribute the origin of anomalous scaling of the velocity instead to the variations of the local energy transfer rate and we shall refer this as Kraichnan's refined similarity idea.

Similar problems of anomalous scaling can be posed for a scalar field advected by a turbulent velocity field. A passive scalar leaves the velocity statistics intact while an active scalar couples with the velocity and influences its statistics. The problem of anomalous scaling of passive scalars is linear and has been recently understood in terms of statistically preserved structures [4]. On the other hand, the nonlinear problem of anomalous scaling of active scalars, like that of velocity, remains unsolved. A common example of an active scalar is temperature in turbulent convection in which temperature variations drive the flow. Turbulent convection is often investigated experimentally in Rayleigh-Bénard convection cells heated from below and cooled on top (see, e.g., [5, 6, 7] for a review). Such confined convective flows are highly inhomogeneous with thermal and viscous boundary layers near the top and the bottom of the cell. Moreover, coherent structures, known as plumes, could affect the scaling properties [8]. For the purpose of studying anomalous scaling of an active scalar, it would thus be more desirable to study homogeneous turbulent convection and in the absence of coherent structures.

Homogeneous turbulent convection has been proposed [9] as a convective flow in a box, with periodic boundary conditions, driven by a constant temperature gradient along the vertical direction. In Boussinesq approximation [10], the equations of motion read [11]:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}p + \nu\nabla^2\vec{u} + \alpha g\theta\hat{z} \qquad (1)$$

$$\frac{\partial \theta}{\partial t} + \vec{u}\cdot\vec{\nabla}\theta = \kappa\nabla^2\theta + \beta u_z \qquad (2)$$

with $\vec{\nabla}\cdot\vec{u} = 0$. Here, $\vec{u}$ is the velocity, p is the pressure divided by the density, $\theta = T - (T_0 - \beta z)$ is the deviation of temperature T from a linear profile of constant temperature gradient of $-\beta$, T_0, α, ν, and κ are respectively the mean temperature, the volume expansion coefficient, kinematic viscosity and thermal diffusivity of the fluid, g is the acceleration due to gravity, and $\hat{z}$ is a unit vector in the vertical direction. The Bolgiano length [12], given by $L_B = \epsilon^{5/4}\chi^{-3/4}(\alpha g)^{-3/2}$, where ϵ and χ are respectively the average energy and thermal dissipation rates, is an estimate of a length scale above which buoyant forces are dominant. Numerical studies [9, 13] revealed that L_B is of the order of the size of the periodic box. As a result, temperature is not active in the intermediate scales. Indeed the small-scale isotropic fluctuations were found [13] to have scaling close to that of K41.

On the other hand, a dynamical shell model for homogeneous turbulent convection driven by a temperature gradient has also been proposed [14]. Shell model is constructed in a discretized Fourier space with $k_n = k_0 h^n$, $n = 0, 1, \ldots, N-1$, being the wavenumber in the nth shell, and h and k_0 are customarily taken to be 2 and 1 respectively. Shell models for homogeneous and isotropic turbulence have been studied extensively and proved to be successful in reproducing the scaling properties ob-

served in experiments [15]. In this shell model for homogeneous turbulent convection, real variables u_n and θ_n representing the Fourier transforms of $\vec{u}$ and θ satisfy the evolution equations:

$$\begin{aligned} \frac{du_n}{dt} + \nu k_n^2 u_n &= a k_n (u_{n-1}^2 - h u_n u_{n+1}) \\ &+ b k_n (u_n u_{n-1} - h u_{n+1}^2) + \alpha g \theta_n \quad (3) \\ \frac{d\theta_n}{dt} + \kappa k_n^2 \theta_n &= \tilde{a} k_n (u_{n-1}\theta_{n-1} - h u_n \theta_{n+1}) \\ &+ \tilde{b} k_n (u_n \theta_{n-1} - h u_{n+1}\theta_{n+1}) + \beta u_n \quad (4) \end{aligned}$$

where a, b, $\tilde{a}$, and $\tilde{b}$ are positive parameters. It was reported that the scaling behavior depends on the ratio b/a [14]: close to Bolgiano-Obukhov (BO) scaling [12, 16] ($u_n \sim k_n^{-3/5}$, $\theta_n \sim k_n^{-1/5}$) for b/a large and close to K41 scaling [1] ($u_n \sim k_n^{-1/3}$, $\theta_n \sim k_n^{-1/3}$) for b/a smaller than about 0.4. In this Letter, we show that buoyant forces are important when b/a is large and insignificant when b/a is small in this shell model of homogeneous turbulent convection and focus on b/a large to study anomalous scaling of an active scalar. We extend refined similarity ideas for homogeneous and isotropic turbulence to homogeneous turbulent convection. Specifically, we extend Kraichnan's refined similarity idea and attribute the origin of the anomalous scaling in homogeneous turbulent convection to variations of the entropy transfer rate. Using numerical simulations of the model, we verify the consequences and thus the validity of our hypothesis.

Multiply Eq. (3) by u_n, we get the energy budget:

$$\frac{dE_n}{dt} = F_u(k_n) - F_u(k_{n+1}) - \nu k_n^2 u_n^2 + \alpha g u_n \theta_n \quad (5)$$

where $E_n = u_n^2/2$ is the energy in the nth shell, $F_u(k_n) \equiv k_n(a u_{n-1} + b u_n) u_{n-1} u_n$ is the rate of energy transfer from $(n-1)$th to nth shell, $\nu k_n^2 u_n^2$ is the rate of energy dissipation in the nth shell due to viscosity, and $\alpha g u_n \theta_n$ is the power injected into the nth shell by the buoyant forces. It is thus reasonable to take buoyancy to be significant in the nth shell if

$$\alpha g |\langle u_n \theta_n \rangle| > \epsilon \equiv \nu \sum_n k_n^2 \langle u_n^2 \rangle \quad (6)$$

where $\langle \ldots \rangle$ is an average over time. Note that for both K41 and BO scaling, $\alpha g \langle u_n \theta_n \rangle = \epsilon$ at $k_n = 1/L_B$. We find that Eq. (6) is satisfied for most of the shells and thus θ_n is active only for b/a is greater than about 2 (see Fig. 1), and that this change coincides with the reported change in scaling behavior discussed above.

As we are interested in the case of an active scalar, we focus on b/a large and study the velocity and temperature structure functions, $S_p(k_n)$ and $R_p(k_n)$:

$$S_p(k_n) \equiv \langle |u_n|^p \rangle \sim k_n^{-\zeta_p} \; ; \; R_p(k_n) \equiv \langle |\theta_n|^p \rangle \sim k_n^{-\xi_p} \quad (7)$$

The scaling exponents ζ_p and ξ_p do not depend on the values of the various parameters as long as b/a is larger than 2. The results reported below are obtained using $a = 0.01, b = 1$, $\beta = 1$, $\tilde{a} = \tilde{b} = 1$, $\alpha g = 1$, $\nu = 5 \times 10^{-17}$, $\kappa = 5 \times 10^{-15}$ and $N = 32$. As can be seen in Fig. 2, both ζ_p and ξ_p deviate respectively from the BO values of $3p/5$ and $p/5$ obtained from dimensional considerations. Thus the active temperature and the velocity in homogeneous turbulent convection have anomalous scaling. We study also the case where the temperature is driven by a large-scale random forcing instead of an imposed linear gradient. In this case, the βu_n term in Eq. (4) is replaced by a random noise acting only in shell $n = 0$. We find exactly the same scaling exponents, supporting the universality of scaling of an active scalar upon different forcing mechanisms [17, 18].

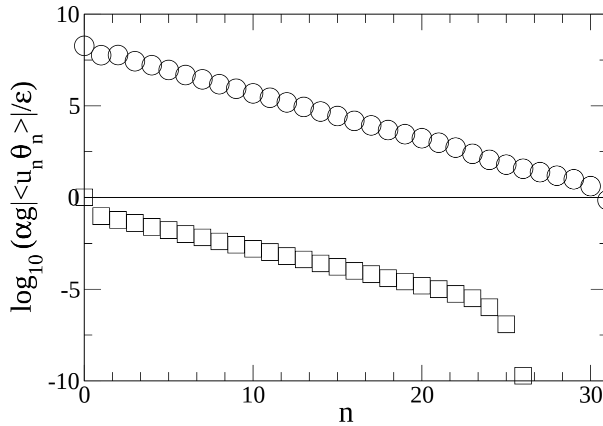

FIG. 1: The logarithm of $\alpha g |\langle u_n \theta_n \rangle|/\epsilon$ for different shells for $a = 0.01, b = 1$, $\beta = 1$, $\nu = 5 \times 10^{-17}$, $\kappa = 5 \times 10^{-15}$ and $N = 32$ (circles), and $a = 10, b = 1$, $\beta = 100$, $\nu = \kappa = 10^{-8}$, and $N = 30$ (squares) (the datapoints for $n = 28$ and 29 are not shown here as they are too small. For both cases, $\tilde{a} = \tilde{b} = 1$ and $\alpha g = 1$.

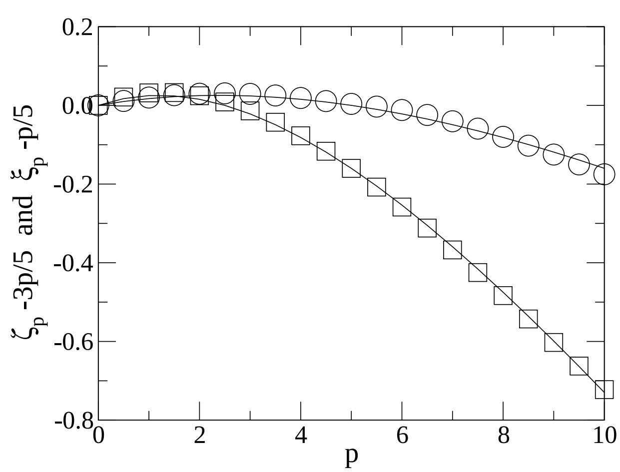

FIG. 2: Deviation of the scaling exponents from the BO values: $\zeta_p - 3p/5$ (circles) and $\xi_p - p/5$ (squares). The solid lines are the results of Eqs. (15) and (16).

It was suggested [19] that when buoyancy is dominant, the scaling behavior of velocity and temperature spectra is governed by an entropy cascade of constant entropy flux. In Bousinessq approximation, the entropy is proportional to the volume integral of temperature fluctuations. Entropy in the nth shell is, therefore, defined as $\mathcal{S}_n \equiv \theta_n^2/2$. By studying the entropy budget obtained from Eq. (4) upon mulitplication by θ_n:

$$\frac{d\mathcal{S}_n}{dt} = F_\theta(k_n) - F_\theta(k_{n+1}) - \kappa k_n^2\theta_n^2 + \beta u_n\theta_n \tag{8}$$

we get the rate of entropy transfer or entropy flux from $(n-1)$th to nth shell as:

$$F_\theta(k_n) \equiv k_n(\tilde{a}u_{n-1} + \tilde{b}u_n)\theta_{n-1}\theta_n \tag{9}$$

In the stationary state and for intermediate scales where scaling is observed, both $\beta\langle u_n\theta_n\rangle$ and $\kappa k_n^2\langle\theta_n^2\rangle$ are negligible such that there is indeed a constant entropy flux with $\langle F_\theta(k_n)\rangle = \chi \equiv \kappa\sum_n k_n^2\langle\theta_n^2\rangle$.

We propose that when buoyancy is significant,

$$u_n = \phi_u(\alpha g)^{2/5}|F_\theta(k_n)|^{1/5}k_n^{-3/5} \tag{10}$$
$$\theta_n = \phi_\theta(\alpha g)^{-1/5}|F_\theta(k_n)|^{2/5}k_n^{-1/5} \tag{11}$$

where ϕ_u and ϕ_θ are dimensionless random variables that are independent of k_n and statistically independent of $F_\theta(k_n)$. The absolute signs are taken because the entropy flux $F_\theta(k_n)$, unlike χ, can assume both positive and negative values. Equations. (10) and (11) are an extension of Kraichnan's refined similarity idea to homogeneous turbulent convection. With Eqs. (10) and (11), we attribute the anomalous scaling behavior of the active temperature and the velocity to the shell-to-shell variations of the entropy transfer rate. An immediate consequence is that the conditional velocity and temperature structure functions at a certain prescribed value x of the entropy transfer rate are given by

$$\langle |u_n|^p \big| F_\theta(k_n) = x\rangle = \langle\phi_u^p\rangle(\alpha g)^{\frac{2p}{5}}x^{\frac{p}{5}}k_n^{-\frac{3p}{5}} \sim k_n^{-\zeta_p^*} \tag{12}$$
$$\langle |\theta_n|^p \big| F_\theta(k_n) = x\rangle = \langle\phi_\theta^p\rangle(\alpha g)^{-\frac{p}{5}}x^{\frac{2p}{5}}k_n^{-\frac{p}{5}} \sim k_n^{-\xi_p^*} \tag{13}$$

and hence would have simple scaling with BO exponents of $\zeta_p^* = 3p/5$ and $\xi_p^* = p/5$ respectively. We evaluate the conditional velocity and temperature structure functions at different values of x and confirm that ζ_p^* and ξ_p^* are independent of x, and in good agreement with $3p/5$ and $p/5$ respectively as shown in Fig. 3.

Let $\langle|F_\theta(k_n)|^p\rangle \sim k_n^{-\tau_p}$, then Eqs. (10) and (11) imply

$$\zeta_p = 3p/5 + \tau_{p/5}\ ; \qquad \xi_p = p/5 + \tau_{2p/5} \tag{14}$$

showing that the intermittency corrections, which are the deviations of the scaling exponents from the BO values,

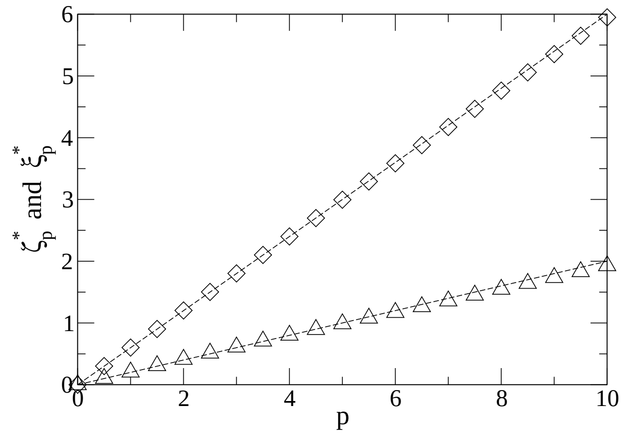

FIG. 3: The scaling exponents ζ_p^* (diamonds) and ξ_p^* (triangles) of the conditional velocity and temperature structure functions. They are in good agreement with the BO values of $3p/5$ and $p/5$ (dashed lines).

are given by the scaling exponents of the moments of the entropy transfer rate. As the power of F_θ in R_p is twice that in S_p, this explains why the anomaly is larger for ξ_p than for ζ_p (see Fig. 2). We evaluate τ_p numerically and check Eq. (14) in Fig. 4. Good agreement is again found.

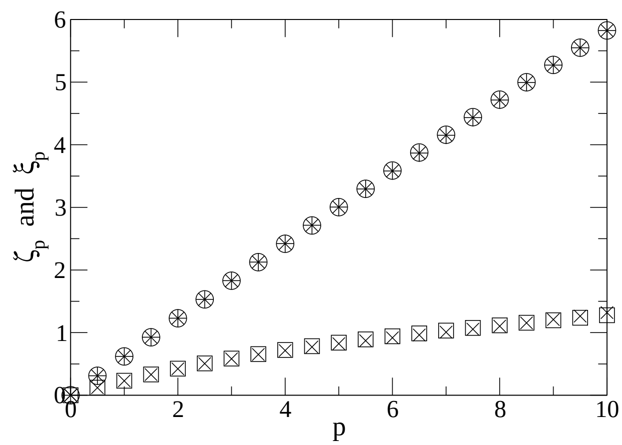

FIG. 4: Comparison of ζ_p (circles) and ξ_p (squares) with the theoretical predictions of $3p/5 + \tau_{p/5}$ (stars) and $p/5 + \tau_{2p/5}$ (crosses) using the numerical results of τ_p.

Next, we show that the intermittency corrections, as given by τ_p, can be obtained by suitably modifying the results of the scaling exponents of the moments of the local thermal dissipation rate found in experiments [20]. In Ref. [20], the statistics of the local thermal dissipation rate, estimated by χ_τ, have been studied in the central region of turbulent Rayleigh-Bénard convection. It was found that the moments of $\chi_\tau \equiv (\langle u_c^2\rangle\tau)^{-1}\int_t^{t+\tau}\kappa(\partial T/\partial t')^2dt'$, where $\langle u_c^2\rangle$ is the mean square velocity fluctuations at the centre, satisfy a hi-

erarchical structure of the She-Leveque form [21], and that their scaling exponents μ_p, defined by $\langle \chi_\tau^p \rangle \sim \tau^{\mu_p}$, can be well described by $\mu_p = c(1-\beta_\chi^p) - \lambda p$ with $c=1$, $\beta_\chi = 2/3$ and $\lambda = 1/3$. The parameter λ is the scaling exponent of $\lim_{p\to\infty} \langle \chi_\tau^{p+1} \rangle / \langle \chi_\tau^p \rangle$, which was estimated [20] as the ratio of the maximum thermal dissipation divided by a time t_r at the scale $r = \sqrt{\langle u_c^2 \rangle}\tau$. Taking t_r as r/u_r, $b = 1/3$ implies $u_r \sim r^{1/3}$, which is Kolmogorov scaling. Here, we find that the moments of the entropy transfer rate $\langle |F_\theta(k_n)|^p \rangle$ also satisfy the same hierarchical structure [22] and similarly τ_p is well approximated by $c_1(1-\gamma^p) - c_2 p$. Similarly, c_2 is the scaling exponent of $F_\theta^{(\infty)}(k_n) \equiv \lim_{p\to\infty} \langle |F_\theta(k_n)|^{p+1} \rangle / \langle |F_\theta(k_n)|^p \rangle$. Following Ref. [20], we estimate $F_\theta^{(\infty)}(k_n)$ as $\mathcal{S}_{max} u_n k_n$, where $\mathcal{S}_{max}$ is the largest possible entropy. Since we observe BO-like scaling in the present case, it is more appropriate to estimate $u_n \sim k_n^{-3/5}$. As a result, we get $c_2 = 2/5$. Then as $\langle |F_\theta(k_n)| \rangle \sim \langle F_\theta(k_n) \rangle = \chi$ is independent of k_n, we have $\tau_1 = 0$ implying $c_1(1-\gamma) = c_2$. If we keep $c_1 = 1$ as $c = 1$ for χ_τ, then we get $\gamma = 3/5$. Putting these results together and using Eq. (14), we find

$$\zeta_p - 3p/5 = 1 - (3/5)^{p/5} - 2p/25 \tag{15}$$

$$\xi_p - p/5 = 1 - (3/5)^{2p/5} - 4p/25 \tag{16}$$

Interesting, as shown in Fig. 2, Eqs. (15) and (16) indeed describe the measured intermittency corrections well.

We have focussed on understanding the origin of anomalous scaling of an active scalar in homogenous turbulent convection using a dynamical shell model. We have extended Kraichnan's refined similarity idea to an active scalar in homogeneous turbulent convection and attributed the anomalous scaling to the variations in the entropy transfer rate. We have verified our hypothesis by showing explicitly that the conditional velocity and temperature structure functions at fixed values of the entropy transfer rate have simple scaling exponents of the BO values, and the intermittency corrections are given by the scaling exponents of the entropy transfer rate. Furthermore, by modifying earlier results obtained for the statistics of the local thermal dissipation rate in turbulent Rayleigh-Bénard convection [20], we have obtained the scaling exponents τ_p of the moments of the entropy transfer rate and thus Eqs. (15) and (16) for the intermittency corrections $\zeta_p - 3p/5$ and $\xi_p - p/5$. These results are found to be in good agreement with the numerical values obtained in the simulations of the shell model.

We should note that the scaling behavior of homogeneous turbulent convection might not be the same as that in the central region of confined turbulent convection as coherent structures present in the latter case could affect the scaling properties [8]. Indeed direct numerical simulations [23] and analyses of experimental data [24] indicated that the scaling behavior of the central region of confined turbulent convection is not well described by BO scaling plus intermittency corrections. On the other hand, there is evidence [25] of the validity of extending Kolmogorov's refined similarity idea in terms of the local thermal dissipation rate in the central region of confined turbulent convection. It would be interesting to further investigate this issue.

We thank TC Ko for his help in plotting Fig. 1. This work is supported in part by the Hong Kong Research Grants Council (Grant No. CUHK 400304).

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